SMOOTH MANIFOLDS FALL 2022 - HOMEWORK 7

Problem 1. Let G be a connected Lie group, and $\mathfrak{h}$ be an ideal of $\mathrm{Lie}(G)$.

(a) Show that there is a well defined action $\overline{\mathrm{Ad}} : G \to GL(\mathrm{Lie}(G)/\mathfrak{h})$ defined by

$$\overline{\mathrm{Ad}}(g)(X + \mathfrak{h}) = \mathrm{Ad}(g)X + \mathfrak{h}.$$

(b) Show that if $g = \exp(X)$ is sufficiently close to $e \in G$, $g \in \ker \overline{\mathrm{Ad}}$ if and only if the image of $\mathrm{ad}(X)$ is contained in $\mathfrak{h}$.

(c) Give an example of a Lie group and proper ideal for which Ad is faithful but $\overline{\mathrm{Ad}}$ is trivial.

Solutions.

(a) To see that $\overline{\mathrm{Ad}}(g)$ is well-defined, we must show that if $Y' \in Y + \mathfrak{h}$, then $\mathrm{Ad}(g)Y' \in \mathrm{Ad}(g)Y + \mathfrak{h}$. Since $\mathrm{Ad}(g)$ is linear, it suffices to show that $\mathrm{Ad}(g)\mathfrak{h} \subset \mathfrak{h}$ for every $g \in G$. Since G is connected, every $g \in G$ can be written as a finite product of elements of the form $\exp(X)$ for $X \in \mathrm{Lie}(G)$. Notice also that

$$\mathrm{Ad}\left(\exp(X_1)\exp(X_2)\ldots\exp(X_n)\right) = \mathrm{Ad}(\exp(X_1))\,\mathrm{Ad}(\exp(X_2))\ldots\mathrm{Ad}(\exp(X_n)).$$

Therefore, it suffices to show that for every $X \in \mathrm{Lie}(G)$, $\mathrm{Ad}(\exp(X))\mathfrak{h} \subset \mathfrak{h}$. Finally, since $\mathrm{Ad}(\exp(X)) = \exp(\mathrm{ad}(X))$, and $\mathrm{ad}(X)$ preserves $\mathfrak{h}$, the claim follows.

(b) First, assume that $\mathrm{ad}(X)\,\mathrm{Lie}(G) \subset \mathfrak{h}$. Then

$$\mathrm{Ad}(g)Y = \exp(\mathrm{ad}(X))Y = Y + \mathrm{ad}(X)Y + \frac{1}{2}\,\mathrm{ad}(X)^2Y + \ldots$$

By asusmption, all terms except the first one belong to $\mathfrak{h}$. Hence $\mathrm{Ad}(g)Y \in Y + \mathfrak{Y}$ for all $Y \in \mathrm{Lie}(G)$. Hence $\overline{\mathrm{Ad}}(g) = \mathrm{Id}$, and $g \in \ker \overline{\mathrm{Ad}}$. Conversely, observe that since $\overline{\mathrm{Ad}}$ is a continuous homomorphism, $\ker \overline{\mathrm{Ad}}$ is a closed subgroup of G. Therefore, it is a closed Lie subgroup, and there exists a neighborhood of $e \in G$ such that if $g \in \ker \overline{\mathrm{Ad}}$, then $g = \exp(X)$ for some $X \in \mathrm{Lie}(G)$, and $g_t = \exp(tX)$ belongs to $\ker \overline{\mathrm{Ad}}$ for all $t \in (-1,1)$. Since $\overline{\mathrm{Ad}}(\exp(tX)) = \mathrm{Id}_{\mathrm{Lie}(G)/\mathfrak{h}}$ for all $t \in (-1,1)$, $\mathrm{Ad}(\exp(tX))Y \in Y + \mathfrak{h}$. Hence,

$$\mathrm{ad}(X)Y = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}(\exp(tX))Y = \lim_{t\to 0} \frac{1}{t}\left(\mathrm{Ad}(\exp(tX))Y - Y\right) \in \mathfrak{h}$$

Since $X \in \mathrm{Lie}(\ker \overline{\mathrm{Ad}})$ and $Y \in \mathrm{Lie}(G)$ were arbitrary, we have finished the proof.

(c) Let $G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a > 0, b \in \mathbb{R} \right\}$, so that $\mathrm{Lie}(G) = \left\{ \begin{pmatrix} s & t \\ 0 & 0 \end{pmatrix} : s, t \in \mathbb{R} \right\} = \mathrm{span}\,\{X, Y\}$, where $X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. It follows that $[X,Y] = Y$. Let $\mathfrak{h} = \mathbb{R} \cdot Y$. Since $[X, Y] = Y$, it follows that $\mathfrak{h}$ is an ideal. Furthermore, for $g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$,

$$\mathrm{Ad}(g)\begin{pmatrix} s & t \\ 0 & 0 \end{pmatrix} = g\begin{pmatrix} s & t \\ 0 & 0 \end{pmatrix}g^{-1} = \begin{pmatrix} s & at - bs \\ 0 & 0 \end{pmatrix}.$$

Therefore, since the image differs from the input by an element of Y $\overline{\mathrm{Ad}}(g) = \mathrm{Id}$ for all g. Since G is center-free, Ad is faithful.

□

Problem 2. If V is a vector space of dimension n and $\omega \in \Lambda^k(V)$ is an alternating k-multilinear function and $v \in V$, let $\iota_v \omega \in \Lambda^{k-1}(V)$ be the functional $(w_1, \ldots, w_{k-1}) \mapsto \omega(v, w_1, \ldots, w_{k-1})$. Let $\ker \omega = \{v \in V : \iota_v \omega = 0\}$.

(a) Show that $\ker \omega$ is a subspace of V.
(b) Prove or find a counterexample: for a fixed v, $\iota_v : \Lambda^2(V) \to \Lambda^1(V)$ is never onto.
(c) Prove or find a counterexample: if $W \subset \ker \omega$, then ω induces a well defined alternating k-multilinear function on V/W.

Solution.

(a) Let $v_1, v_2 \in \ker \omega$. Then for every $(w_1, \ldots, w_{k-1}) \in V^{k-1}$,

$$\begin{aligned}
\iota_{v_1+cv_2}\omega(w_1, \ldots, w_{k-1}) &= \omega(v_1 + cv_2, w_1, \ldots, w_{k-1}) \\
&= \omega(v_1, w_1, \ldots, w_{k-1}) + c\omega(v_2, w_1, \ldots, w_{k-1}) \\
&= 0 + c \cdot 0 \\
&= 0
\end{aligned}$$

Hence, $\ker \omega$ is a subspace of V.

(b) This is true. Indeed, fix a v, and notice that for any $\omega \in \Lambda^2(V)$, $\iota_v \omega(v) = \omega(v, v) = 0$. In particular, $v \in \ker \iota_v \omega$, so any $\theta \in \Lambda^1(V) = V^*$ such that $\theta(v) = 1$ cannot be in the image of ι_v.

(c) This is true. We define $\bar{\omega} \in \Lambda^k(V/W)$ by

$$\bar{\omega}(v_1 + W, \ldots, v_k + W) = \omega(v_1, \ldots, v_k).$$

To see that this is well-defined, it suffices to show that for any $(w_1, \ldots, w_k) \in W^k$, $\omega(v_1 + w_1, \ldots, v_k + w_k) = \omega(v_1, \ldots, v_k)$. Define $z_{i,0} = v_i$ and $z_{i,1} = w_i$, so that

$$\omega(v_1 + w_1, \ldots, v_k + w_k) = \sum_{\sigma \in \{0,1\}^k} \omega(z_{1,\sigma_1}, \ldots, z_{k,\sigma_k})$$

Unless $\sigma = (0, \ldots, 0)$, at least one of the terms x_{i,σ_i} is equal to w_i. Since $w_i \in W \subset \ker \omega$, using the antisymmetry of ω, it follows that the term vanishes. Hence the only term which survives is the $\sigma = (0, \ldots, 0)$ term. Hence,

$$\omega(v_1 + w_1, \ldots, v_k + w_k) = \omega(v_1, \ldots, v_k),$$

and $\bar{\omega}$ is well-defined on V/W.

□

Problem 3. Let V be an n-dimensional vector space and $\omega \in \Lambda^k(V)$. If $W \subset V$ is a vector subspace, let $\pi_W : \Lambda^k(V) \to \Lambda^k(W)$ denote the restriction map, $\pi_W(\alpha) = \alpha|_W$.

(a) Show that π_W is linear and onto. Use this to compute $\dim(\ker \pi_W)$.
(b) Find a basis for $\ker \pi_W$ when $V = \mathbb{R}^5$, $W = \mathbb{R}^3 \times \{0\}$ and $k = 2$.

Solution.

(a) Linearity follows immediately, since $\pi_W(\alpha + c\beta)(w) = (\alpha + c\beta)|_W(w_1, \ldots, w_k) = \alpha(w_1, \ldots, w_k) + c\beta(w_1, \ldots, w_k) = \pi_W(\alpha)(w_1, \ldots, w_k) + c\pi_W(\beta)(w_1, \ldots, w_k)$. To see that it is onto, choose a subspace $W' \subset V$ such that $V = W \oplus W'$, and $p : V \to W$ be the corresponding linear projection sending $w + w' \mapsto w$. If $\omega \in \Lambda^k(W)$, define $\tilde{\omega} \in \Lambda^k(V)$ by $\tilde{\omega}(v_1, \dots, v_k) = \omega(p(v_1), \dots, p(v_k))$ (ie, $\tilde{\omega} = p^*\omega$). It is clear that $\pi_W \tilde{\omega} = \omega$, so π_W is onto.

(b) A basis can be constructed as follows:

$$\{dx_1 \wedge dx_4, dx_2 \wedge dx_4, dx_3 \wedge dx_4, dx_1 \wedge dx_5, dx_2 \wedge dx_5, dx_3 \wedge dx_5, dx_4 \wedge dx_5\}$$

Indeed, these are $7 = \binom{5}{2} - \binom{3}{2}$ linearly independent elements of $\Lambda^2(\mathbb{R}^5)$, and each of them becomes 0 on $\mathbb{R}^3$ since all of them involve at least one of dx_4 or dx_5, which evaluate to 0 on $\mathbb{R}^3$.

$\square$